

The limit configuration space integral for tangles and the Kontsevich integral

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Abstract

This article is the continuation of our first article (math/9901028). It shows how the zero-anomaly result of Yang implies the equality between the configuration space integral and the Kontsevich integral.

1 Introduction

The construction of the configuration space integral for knots that we made in our previous article [P] admits a straightforward generalisation to links and tangles. We will not detail this construction again, but we will make use of most of it to prove the following theorem:

Theorem 1.1 *The configuration space integral and the Kontsevich integral coincide on the isotopy classes of framed links.*

This framing is defined differently in each case: for the configuration space integral, it is equal to the Gauss integral $I_K(\theta)$ (so it can vary continuously), but its limit value (in the sense below) coincides with the value of the framing in the Kontsevich integral.

Let us redefine the configuration space integral, now generalised to any one-dimensional compact manifold with boundary:

Let M be a compact one-dimensional manifold with boundary. We denote by L an imbedding of M into \mathbb{R}^3 .

Definition 1.2 *A diagram with support M will be a graph made of a finite set $V = U \cup T$ of vertices and a set E of edges which are pairs of elements of V , together with an injection i of U into the interior of M up to isotopy, such that: the elements of U are univalent and the elements of T are trivalent, and every connected component of this diagram meets U . The set of these diagrams up to isomorphism is denoted by \mathcal{D}_M .*

Definition 1.3 A configuration of a diagram $\Gamma \in \mathcal{D}_M$ on L is a map from the set V of vertices of Γ to \mathbb{R}^3 , which is injective on each edge, and which coincides on U with $L \circ i$ for some i in the class. The configuration space $C_\Gamma(L)$ of a diagram $\Gamma \in \mathcal{D}_M$ on L is the set of these configurations.

The orientation of $C_\Gamma(L)$ is defined in the same way as in Subsection 2.2 of our previous article (it depends on an arbitrary choice of orientations of all edges and trivalent vertices, and it is reversed when we change exactly one of them), except that here we need to define it without orienting M . To do it, we give a local orientation of M at the position of each univalent vertex (whereas these local orientations were all equal to the orientation of $M = S^1$). Then, we will have the antisymmetry relation also for univalent vertices.

When orientations of all edges are chosen, we have a canonical map Ψ from $\overline{C}_\Gamma(L)$ to $(S^2)^E$ (where E is the set of edges of Γ).

Definition 1.4 We define the space $\mathcal{A}(M)$ to be the real vector space generated by the set \mathcal{D}_M of diagrams with support M with orientations on the trivalent vertices, modulo the AS, IHX and STU relations, and completed with respect to the degree (where the degree of a diagram is half its number of vertices).

This space $\mathcal{A}(M)$ has the structure of a cocommutative coalgebra, with also a unit which is the class of the empty diagram.

Definition 1.5 We define the integral

$$Z(L) = \sum_{\Gamma \in \mathcal{D}_M} \frac{I_L(\Gamma)}{|\Gamma|} [\Gamma] \in \mathcal{A}(M),$$

where:

$[\Gamma]$ is the image of Γ in $\mathcal{A}(M)$;

$|\Gamma|$ is the number of automorphisms of Γ (considered without vertex-orientations);

and

$$I_L(\Gamma) = \int_{C_\Gamma(L)} \Psi^* \left(\bigwedge^E \omega \right)$$

where ω denotes the standard volume form on S^2 with total mass 1.

We have proved in our previous article:

- (Lemma 2.11) We can restrict ourselves to diagrams which are triply connected to U (see Definition 2.5). (the integrals for other diagrams vanish).
- For each integer n , the degree n part of Z can be viewed in some way as the integral of the pullback of Ψ over a formal single manifold which is a glued union of configuration spaces compactifications of diagrams with coefficients in the degree n part of $\mathcal{A}(M)$. In the case of links, the only remaining possibly *non-degenerate* faces of this space (i.e. faces whose images by Ψ have codimension only one) are the faces of the anomaly, that is, the sets of limit configurations

obtained when a connected component of a diagram collapses to a point of the knot.

- (Proposition 4.4) $Z(L)$ is a grouplike element of the coalgebra $\mathcal{A}(M)$.
- (Corollary 7.2) On each isotopy class of knots, Z is a function of $I_K(\theta)$ of the form

$$Z = Z_0 \exp \left(\frac{I_K(\theta)}{2} a \right)$$

where Z_0 is an invariant of the knot K , and a is the value of the ‘anomaly’: it is a constant element of $\mathcal{A}(S^1)$ which does not depend on K (see Definition 6.3 of [P]) and is equal to $[\theta]$ according to [Ya].

We recall (see Subsections 3.1 and 3.2) that given a graph G (made of the same set V of vertices as the diagram we consider, and at least the same set of edges), we compactify the space of its configurations modulo dilations and translations, by imbedding it into the space \mathcal{H} which is the product of the C^A ’s for all the connected parts A of V with cardinality at least two, where C^A denotes the space $(\mathbb{R}^3)^A / TD$ of nonconstant maps from A to \mathbb{R}^3 quotiented by the translation-dilations group.

2 Sketch of the proof

Definition 2.1 *The compactification $\overline{C}_\Gamma(L)$ of this configuration space is defined to be its adherence in the compact manifold $\mathcal{H}' = \mathcal{H} \times M^U$ where it is embedded, where, in the graph G used to define \mathcal{H} , we let all pairs of univalent vertices be edges: thus U is connected in this graph and the corners have the same form as with knots.*

The same constructions work, except that when M has a boundary, there are some more variations of $Z(L)$ during isotopies, coming from a new boundary of the compactified configuration space $\overline{C}_\Gamma(L)$, made of the configurations in which some univalent vertex reaches the boundary of L .

For $\lambda \in \mathbb{R}$, let H_λ be the following map:

$$\begin{aligned} H_\lambda : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x_1, x_2, x_3) &\longmapsto (\lambda x_1, \lambda x_2, x_3) \end{aligned}$$

Now let $L_\lambda = H_\lambda \circ L$. We are going to study the limit of a space $C_\Gamma(L_\lambda)$, viewed as a submanifold of \mathcal{H}' (or more precisely a submanifold of $\overline{C}(G) \times M^U$) when λ approaches zero.

For technical reasons, we shall suppose in all the following that all the imbeddings L that we will consider are algebraic imbeddings (this means for example that the restriction of L to a loop component of M has a finite Fourier series), and that the map L_0 , as a map from M to the vertical line $\{0\}^2 \times \mathbb{R} \approx \mathbb{R}$ of \mathbb{R}^3 , is a Morse map.

We shall prove the following facts:

Proposition 2.2 *When λ approaches zero, the submanifold $C_\Gamma(L_\lambda)$ of \mathcal{H}' has a limit $C_\Gamma^l(L)$ which can be defined as the intersection of $\{0\} \times \mathcal{H}'$ with the adherence of the set of $(\lambda, x) \in]0, 1] \times \mathcal{H}'$ such that $x \in C_\Gamma(L_\lambda)$. On $C_\Gamma^l(L)$, the integral of $\Psi^*\Omega$ converges and is equal to the limit of the integrals of $\Psi^*\Omega$ on the $C_\Gamma(L_\lambda)$ s when λ approaches zero.*

So the limit value $Z^l(L)$ of $Z(L_\lambda)$ is the integral defined on the limits $C_\Gamma^l(L)$ of the configuration spaces C_Γ .

Proposition 2.3 *Fix a diagram Γ . Except for a part of $C_\Gamma^l(L)$ whose image by Ψ has no interior points, the elements $x = (f, f') \in C_\Gamma^l(L)$ verify the following conditions: for the set U_1 of univalent vertices of any connected component of Γ we have*

- 1) $L_0 \circ f'(U_1)$ is a singleton.
- 2) If $L_0 \circ f'(U_1)$ is a critical value of L_0 then $f'(U_1)$ is reduced to the critical point of L_0 .
- 3) If $L_0 \circ f'(U_1)$ is not a critical value of L_0 then $f'(U_1)$ has at least two elements.
- 4) If the univalent vertices of two connected components of Γ are mapped by $L_0 \circ f'$ to the same point, then this point is a critical value of L_0 .

We say that the imbedding L of M into \mathbb{R}^3 is a *tangle* if its image is contained in some $\mathbb{R}^2 \times [a, b]$, the image of the boundary $L(\partial M)$ is contained in $\mathbb{R}^2 \times \{a, b\}$ and the tangent vectors at this boundary are not horizontal. For any $c \in]a, b[$, we say that L is the *product* $L = L' \cdot L''$ where L' and L'' are the respective restrictions of L on the parts M' and M'' such that $M = M' \cup M''$ with images in $[c, b]$ and $[a, c]$ respectively. Here, if c is a critical value of L_0 , then the corresponding critical point will belong to the one part among M' and M'' in which it is interior. We have a canonical graded bilinear product from $\mathcal{A}(M') \times \mathcal{A}(M'')$ to $\mathcal{A}(M)$.

Proposition 2.4 *With the above notations we have*

$$Z^l(L) = Z^l(L') \cdot Z^l(L'').$$

Definition 2.5 *Let L and L' be two tangles defined on M which coincide on ∂M (we suppose the interval $[a, b]$ is fixed). An isotopy between L and L' is a smooth map ϕ from $[0, 1] \times M$ to \mathbb{R}^3 such that $L = \phi(0, \cdot)$, $L' = \phi(1, \cdot)$, and $\forall t \in [0, 1]$, $\phi(t, \cdot)$ is a tangle which coincides with L on ∂M , except that $H_0(\phi(t, \cdot))$ is not necessarily a Morse function for $t \in]a, b[$.*

Proposition 2.6 *If there is an algebraic isotopy ϕ between L and L' such that the set of values of the horizontal tangent vectors to all tangles $\phi(t, \cdot)$ is finite (in the set of horizontal directions), then $Z^l(L) = Z^l(L')$.*

Notations 2.7 Let J be a finite set with at least two elements. We denote by \mathcal{C}_J the space of injections of J into the plane \mathbb{R}^2 modulo dilations and translations. Let $M_J = J \times \mathbb{R}$ (this is equivalent to taking a compact manifold: just compactify \mathbb{R} as a segment). The space $\mathcal{A}(M_J)$ (denoted in short by $\mathcal{A}(J)$) has the structure of a non-commutative Hopf algebra.

Let π be the canonical map from the set U of univalent vertices of a $\Gamma \in \mathcal{D}(M_J) = \mathcal{D}(J)$ to J .

Let \mathcal{D}_J^p be the set of the nonempty connected diagrams in $\mathcal{D}(J)$ on which π is not constant.

Let $\mathcal{P}(J)$ be the closed subspace of $\mathcal{A}(M_J) = \mathcal{A}(J)$ which is the product of the spaces generated by the elements of \mathcal{D}_J^p at each degree. This space $\mathcal{P}(J)$ is stable under the Lie bracket $[X, Y] = X \cdot Y - Y \cdot X$, thus it is a Lie algebra.

For a configuration $x \in \mathcal{C}_J$, we define the embedding $x \times \text{Id}_{\mathbb{R}}$ of $M_J = J \times \mathbb{R}$ into $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$. For any $\Gamma \in \mathcal{D}_J^p$, let $C_\Gamma(x)$ be the configuration space $C_\Gamma(x \times \text{Id}_{\mathbb{R}})$ modulo the vertical translations.

Now define the space C_Γ to be the total space of a fibration over \mathcal{C}_J such that the fiber over $x \in \mathcal{C}_J$ is $C_\Gamma(x)$. We still define the canonical map Ψ from C_Γ to $(S^2)^E$.

Note that the dimension of each fiber $C_\Gamma(x)$ is one less than the dimension of the target space $(S^2)^E$ of Ψ , because we quotiented by the vertical translations. Thus the integration of $\Phi^* \Omega$ along the fibers of $C(\Gamma)$ gives a 1-form ω_Γ on the manifold \mathcal{C}_J . But we must specify the orientations to define this form completely: we define the orientation of $C_\Gamma(x \times \text{Id}_{\mathbb{R}})$ in the same way as in Subsection 2.2 of our previous article, and we substitute the dimension of a tangent vector to \mathcal{C}_J , to the dimension of the vertical translations by which we quotient.

Definition 2.8 With the above notations, let $\omega(J)$ be the 1-form on \mathcal{C}_J with values in $\mathcal{P}(J)$ defined by

$$\omega(J) = \sum_{\Gamma \in \mathcal{D}_J^p} \frac{\omega_\Gamma[\Gamma]}{|\Gamma|}.$$

Recall that a braid is a tangle with no horizontal tangent vector, and it can also be seen as a path in \mathcal{C}_J . We shall see the following proposition:

Proposition 2.9 $\omega(J)$ is a flat connection over \mathcal{C}_J , and the above integral $Z^l(\mathcal{B})$ for a braid \mathcal{B} is equal to the monodromy of the connection $\omega(J)$ along the path \mathcal{B} .

Now we are going to compactify the space \mathcal{C}_J . If $B \subseteq J$, let \mathcal{C}'_B be the space of nonconstant maps from B to the plane \mathbb{R}^2 modulo the dilations and translations. It is a compact manifold. We have a canonical imbedding:

$$\mathcal{C}_J \hookrightarrow \prod_{B \subseteq J, \#B \geq 2} \mathcal{C}'_B.$$

This provides a compactification $\overline{\mathcal{C}}_J$ of \mathcal{C}_J .

A stratum of this compactification is labelled by a set S of parts of J such that $J \in S$, any B in S has at least two elements and any two elements of S are either disjoint or one included in the other. It is parametrised by the space

$$\mathcal{C}_{J,S} = \prod_{B \in S} \mathcal{C}_{B/S}$$

where B/S is the quotient set of B by the greatest elements of S strictly included in B . (See Subsection 3.1 of [P] for more details.)

We define the fibered space $\overline{\mathcal{C}}_\Gamma$ over $\overline{\mathcal{C}}_J$ to be the closure of \mathcal{C}_Γ in the compact space $\overline{\mathcal{C}}_J \times \mathcal{H}'$. This fibered space $\overline{\mathcal{C}}_\Gamma$ is not locally trivial near the boundary of $\overline{\mathcal{C}}_J$ (some singularities may appear).

Next we are going to express the restriction $\omega(J, S)$ of the connection $\omega(J)$ to $\mathcal{C}_{J,S}$ in terms of the connections $\omega(B/S)$. This restriction is defined by integration along the fibers of the fibered space which is the restriction of $\overline{\mathcal{C}}_\Gamma$ over $\mathcal{C}_{J,S}$.

Definition 2.10 *Define the duplication map $\delta : \mathcal{A}(B/S) \longrightarrow \mathcal{A}(B) \subseteq \mathcal{A}(J)$ in the following way: for each diagram $\Gamma \in \mathcal{D}(B/S)$, consider the set of maps f from the set of univalent vertices of Γ to B such that f composed with the canonical map from B to B/S is the map π of Γ . This f defines a diagram in $\mathcal{D}(J)$ by preserving the linear order restricted to each line.*

We define $\delta(\Gamma)$ to be the sum of these diagrams running over the set of such maps f .

It is a common thing to check that for any fixed S , the images by δ of the $\mathcal{A}(B/S)$ for the different $B \in S$ commute in $\mathcal{A}(J)$ (it is the same proof as the proof that the product is well-defined in $\mathcal{A}(S^1)$ [BN]).

Proposition 2.11 $\omega(J, S) = \prod_{B \in S} \delta(\omega(B/S))$

Proposition 2.12 *For all Γ , the integral of ω_Γ along an algebraic path in $\overline{\mathcal{C}}_J$ converges.*

Consequently, we can define an associator Φ in a natural way:

Definition 2.13 *We define the elements $\Phi \in \mathcal{A}(\{1, 2, 3\})$ and $R \in \mathcal{A}(\{1, 2\})$ by*

$$\Phi = Z^l \left(\left| \begin{array}{c} \diagup \\ \diagdown \end{array} \right| \right) \quad , \quad R = Z^l \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right).$$

Here in the definition of Φ , the braid reaches the boundary of $\mathcal{C}_{\{1,2,3\}}$ at each end.

Proposition 2.14 *These elements R and Φ together verify the axioms of the associator.*

Now we have to calculate the value of R , and show that it is equal to $\exp(H/2)$, where H is the diagram in $D_{\{1,2\}}^p$ with only one chord and the same vertical orientation. This is the consequence of the zero-anomaly result of Yang [Ya] and the following formula:

Proposition 2.15 *Let a_1, a_2 be the anomaly with support the first and second component of $M_{\{1,2\}}$ respectively. Then we have*

$$R = \exp \frac{\delta(a_{\{1,2\}}) - a_1 - a_2}{4}.$$

Conclusion. We deduce from 2.15 that the zero-anomaly result $a = [\theta]$ implies $R = \exp(H/2)$. Moreover, Proposition 2.3 2) implies that the integral at the thin slice containing an extremum of a tangle L is the inclusion of

$$Z^l \left(\bigcap \right).$$

According to Theorem 8 of [LM], all invariants of framed oriented links constructed from associators are equal. This implies that the expression Z^l , when applied to a link L such that L_0 is a Morse function and the horizontal tangent vectors to L are parallel, is equal to the Kontsevich integral. But Z^l is just a limit value of Z in an isotopy.

A straightforward generalisation of the formula of the variations of Z in Corollary 7.2 of our previous article is that the variations of Z in an isotopy are expressed in terms of the anomaly and the Gauss integral of each component of the link. So, the configuration space integral and the Kontsevich integral are the same function of the framed links, except that the “framing” (which is a function of the imbedding and may vary continuously) has not the same values in the generic cases but only in the limit cases, in particular for the case of almost planar links.

3 Proofs of the propositions

3.1 A general argument with dimensions

In the proofs of the propositions, we will use several times the following argument to minorate the codimension of the image by Ψ of certain limit parts of the configuration spaces.

Let us consider a connected diagram made of two disjoint finite sets T' and U' of vertices such that $\#U' > 1$, and a set E' of edges which are oriented pairs of elements of $T' \cup U'$. Let L be a one-dimensional submanifold of \mathbb{R}^3 , and n_L

be the dimension of the group \mathcal{G} of dilations-translations of \mathbb{R}^3 which preserve L . For any $x \in T' \cup U'$, let n_x be the number of edges which contain x .

Now we consider the configuration space C consisting of maps from $T' \cup U'$ to \mathbb{R}^3 which map U' to L . Then the codimension of the image of the canonical map Ψ from C to $(S^2)^{E'}$ is minorated by

$$\begin{aligned} \dim(S^2)^{E'} - \dim(C/\mathcal{G}) &= 2\#E' + n_L - 3\#T' - \#U' \\ &= n_L + \sum_{x \in T'} (n_x - 3) + \sum_{x \in U'} (n_x - 1). \end{aligned}$$

This remark will be applied to the case of a connected diagram, (and thus also to any connected component of a diagram): let Γ be a connected diagram with a set U of univalent vertices and a set T of trivalent vertices, and $V = U \cup T$. Let $f \in \mathcal{H}$ be some limit configuration of this diagram. Then f belongs to some stratum of \mathcal{H} defined by a set S of parts of V . Now, apply the above construction to the configuration f_V of the diagram V/S . The images by f_V of the elements of U will be naturally constrained to lay on some one-dimensional manifold L , and since Γ is connected, their images in V/S will be at least univalent. Moreover, the other vertices of V/S (those included in T) will be at least trivalent because of the general assumption that the diagrams are triply connected to U . So, in the above formula we have $n_x - 3 \geq 0$ for any $x \in T'$ and $n_x - 1 \geq 0$ for any $x \in U'$.

If $\#U' = 1$, there are two parameters of variations in the fibers of Ψ in C (the translations which move the element of U' along L , and the dilations with center this element), so we can minorate the codimension of the image of Ψ by

$$2 + \sum_{x \in T'} (n_x - 3) + \sum_{x \in U'} (n_x - 1) \geq 2,$$

which will be enough for our needs.

3.2 The first convergence result

Proof of Proposition 2.2

We have supposed that the embedding L is algebraic (its coordinates are algebraic maps). So the set of $(\lambda, x) \in]0, 1] \times \mathcal{H}'$ such that $x \in C_\Gamma(L_\lambda)$ is an algebraic part of the compact algebraic manifold $[0, 1] \times \mathcal{H}'$. This implies the convergence result and the fact that the limit of the integral is the integral on the limit, by the Stokes theorem. \square

3.3 The separation results

Proof of Proposition 2.3

Consider the set U_1 of univalent vertices of a connected component Γ_1 of Γ . We can suppose that \bar{U}_1 (Notation 3.3) contains Γ_1 , for the type (b) faces were degenerate (Lemma 4.6).

Proof of 1) and 3). Let $(f, f') \in C_\Gamma^l(L)$. We are going to prove that if $L_0 \circ f'(U_1)$ has at least two elements, or $f'(U_1)$ is a singleton which is not a critical point of L_0 , then $\Psi(f)$ lies in a space with codimension at least two (which is independent of f).

Note that the configuration $f_{\bar{U}_1}$ restricted to Γ_1 maps all elements of U_1 to the same vertical line, then conclude with the general argument (3.1). \square

Note that when the singleton $L_0 \circ f'(U_1)$ is not a critical value of L_0 , then the vertical projection of f restricted to U_1 on the horizontal plane coincides with $L_0 \circ f'$ up to translations and dilations. So, $f_{\bar{U}_1}$ maps all elements of U_1 to the finite union of vertical lines which is the preimage under the vertical projection onto the horizontal plane with altitude $L_0 \circ f'(U_1)$, of the intersection of $L(M)$ with this plane.

Proof of 2). If $L_0 \circ f'(U_1)$ is a critical value of L_0 and $f'(U_1)$ is not reduced to the critical point of L_0 , then the configuration $f_{\bar{U}_1}$ restricted to Γ_1 must be considered modulo the vertical translations; it maps all elements of U_1 to the finite union of vertical lines which is the preimage of $L(M)$ by the vertical projection onto the horizontal plane with altitude $L_0 \circ f'(U_1)$. But there are finitely many critical values of L_0 , so $\Psi(f)$ lies in a codimension 1 space.

Proof of 4). If it was not a critical value, then the configuration must be considered modulo independent vertical translations of the two components, which are compensated by the only one parameter of variation $L_0 \circ f'(U_1)$. \square

3.4 The multiplicativity of Z^l

Proof of Proposition 2.4

First consider the tangle as cut into separate parts by the plane $\mathbb{R}^2 \times \{c\}$ (although some ends coincide). Then we deduce that $Z^l(L) = Z^l(L') \otimes Z^l(L'')$ from Proposition 2.3 1) with the same method as the proof that Z is grouplike (Proposition 4.4 of our previous article). Then, glue the ends at the plane $\mathbb{R}^2 \times \{c\}$ to obtain the proposition. \square

3.5 The isotopy invariance

Proof of Proposition 2.6

Remember that the expression Z_n was defined as the integral of $\Psi^*\Omega$ over the formal linear combination of spaces

$$C(L) = \sum_{\Gamma \in D_{n,3}} \alpha(\Gamma) C_\Gamma^l(L).$$

The boundary of this space is made of the faces of the anomaly, the faces corresponding to a univalent vertex meeting a boundary of M , and some degenerate

faces.

Now, for each $\lambda \in]0, 1]$, define the fibered space \mathcal{F}_λ with base $[0, 1]$, and where the fiber over each $t \in [0, 1]$ is the space $C(H_\lambda(\phi(t, \cdot)))$. The canonical map Ψ is well-defined and smooth on \mathcal{F}_λ , so we can apply the Stokes theorem to the closed form $\Psi^*\Omega$: the sum of integrals of $\Psi^*\Omega$ on the faces of \mathcal{F}_λ cancel.

Now, the limits of these integrals when λ approaches 0 are the integrals over the limits of these spaces. This makes sense because we have supposed the isotopy ϕ to be algebraic. So, the Stokes theorem passes to the limit and gives a relation between the integrals over the limits of the different faces of \mathcal{F}_λ . Let us study these limits.

First, we have the two faces defined by $t = 0$ and $t = 1$: they are precisely $C(H_\lambda(\phi(0, \cdot)))$ and $C(H_\lambda(\phi(1, \cdot)))$. So, the integrals on their limits are $-Z_n^l(L)$ $Z_n^l(L')$. To prove that $Z_n^l(L) = Z_n^l(L')$, we just have to show that the form $\Psi^*\Omega$ vanishes on the limits of the other faces.

The degenerate faces do not contribute, because the image by Ψ of their union over t is a one-parameter union of codimension 2 spaces, so it is a codimension 1 space, and the limit of a codimension 1 space still has codimension (at least) one.

In the faces of the anomaly, a set of edges takes a position in $\Psi(W_x(\gamma))$ for some γ and some tangent vector x to L . But $\Psi(W_x(\gamma))$ is independent of L and has codimension 2, so we just have to check that the two-dimensional set of tangent vectors to all the tangles $H_\lambda(\phi(t, \cdot))$ for variable t converges to a dimension 1 limit in S^2 when λ approaches zero. But the non-vertical limit tangent vectors come from the horizontal tangent vectors in the tangles $\phi(t, \cdot)$ (here we use the fact that M is compact): the longitude of a limit tangent vector is given by the value of the corresponding horizontal tangent vector of $\phi(t, \cdot)$. The set of values of the horizontal tangent vectors is finite by assumption, so the limit set of the tangent vectors is contained in the union of the poles and a finite set of meridians. This proves the cancellation of the integral on the faces from the anomaly.

Now, let us check that the integral cancels for the limits (f, f') of configurations in which a univalent vertex reaches the boundary of M . We have to distinguish two cases.

First, suppose that all univalent vertices of the same connected component Γ_1 of Γ are mapped to the same altitude. Then the configuration of Γ_1 belongs to (the closure of) $C_{\Gamma_1}(x)$ where x is the configuration of the boundary points of L in one of the planes $\mathbb{R}^2 \times \{a\}$ and $\mathbb{R}^2 \times \{b\}$.

Second, suppose that not all univalent vertices of Γ_1 are mapped to the same altitude. Then, we conclude with the argument of the proof of Proposition 2.3 1). \square

3.6 The monodromy

Proof of Proposition 2.9

The fact that the integral Z^l of a braid coincides with the expression of the monodromy of the connection ω is an easy consequence of Proposition 2.3 when we view Z and Z^l as an integral over the “knot graphs” as in the proof that Z is grouplike (Proposition 4.4 of our previous article). Of course, here the word is not accurate because it is generalised to braids, but the idea is the same.

Next, we deduce the fact that this connection is flat from Proposition 2.6. \square

3.7 Decomposition of the connection at the boundary

Proof of Proposition 2.11

First, note that for each $\Gamma \in \mathcal{D}(B/S)$ and $x \in \mathcal{C}'_B$, the space of graphs isomorphic to Γ with support $x(B/S) \times \mathbb{R}$ and with coefficient $\delta(\Gamma)$ belongs to the configuration space of the expression of $\omega(J, S)$. Now, what we have to check is that the integrals over the other parts of the limit space which constitutes $\omega(J, S)$ vanish.

These other parts are the ones where the trivalent vertices run through different scales in S . It is easy to see that $\Psi^*\Omega$ vanishes there unless each image point of a univalent vertex in the configuration f_V is connected to the rest of the diagram by only one edge. In this case, if the subdiagram mapped to such a point is non-trivial, it must contain one bivalent vertex. Then, the central symmetry of this bivalent vertex relatively to the middle of the two vertices connected to it (as in the proof of Lemma 5.4 in our previous article), preserves the structure of the graph and reverses the orientation.

So, the integrals on such configurations globally cancel. \square

3.8 Construction of the associator

Proof of Proposition 2.12

This integral is the integral of $\Psi^*\Omega$ on the submanifold of $\overline{\mathcal{C}}_\Gamma$ made of the fibers over the path: this is an algebraically defined manifold, the map Ψ is algebraic and the form $\Psi^*\Omega$ is a smooth form defined on a product of spheres which is a compact manifold, so the integral of $\Psi^*\Omega$ on it converges. \square

Proof of Proposition 2.14.

Let us recall the axioms of an associator (see [LM], section 4, with the notations of Section 1 before Proposition 2):

$$\begin{aligned} (\text{Id} \otimes \text{Id} \otimes \delta)(\Phi) \times (\delta \otimes \text{Id} \otimes \text{Id})(\Phi) &= (1 \otimes \Phi) \times (\text{Id} \otimes \delta \otimes \text{Id})(\Phi) \times (\Phi \otimes 1) \\ (\delta \otimes \text{Id})(R) &= \Phi^{312} \times R^{13} \otimes (\Phi^{132})^{-1} \times R^{23} \times \Phi \\ \Phi^{-1} &= \Phi^{321} \end{aligned}$$

$$\varepsilon_1(\Phi) = \varepsilon_2(\Phi) = \varepsilon_3(\Phi) = 1$$

The first two ones are a direct consequence of Propositions 2.4, 2.6, and 2.11. The third one uses the invariance of Z during the symmetry around a vertical axis (which comes from the invariance of the volume form of S^2 we used to define Z). The last one results from the fact that the braid which defines Φ becomes the identity path in $\mathcal{C}_{1,2}$, $\mathcal{C}_{1,3}$ and $\mathcal{C}_{2,3}$ when we delete the other strand. \square

3.9 Expression of the twist

Proof of Proposition 2.15

Lemma 3.1

$$1_\cap \times R = \exp\left(-\frac{a}{2}\right)$$

Proof. We have

$$Z^l\left(\begin{array}{c} \bigcirc \\ \bigcirc \end{array}\right) = Z^l(\cap)^2(1_\cap \times R)$$

because of the commutativity of the algebra structure of $\mathcal{A}(S^1)$. But we know that it is of the form

$$Z(O) \exp(\mu a)$$

for some $\mu \in \mathbb{R}$, where O is the circle (according to Corollary 7.2 of our previous article). We have obviously

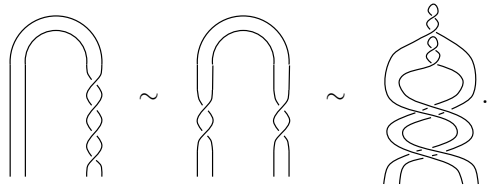
$$Z(O) = Z^l(\cap)^2.$$

Thus,

$$1_\cap \times R = \exp(\mu a)$$

for some μ . So we just have to calculate μ . In this computation, we have to take care of the orientation of the one-dimensional manifold: first, we easily compute that the degree one part of R is $\frac{H}{2}$. Then, we have here a minus sign from the fact that the orientation of one univalent vertex changes from $\begin{array}{c} \nearrow \\ \searrow \end{array}$ to $\begin{array}{c} \searrow \\ \nearrow \end{array}$. \square

Now, consider the following isotopy:



According to Proposition 2.6 and the above lemma, we have

$$Z^l \left(\text{diagram 1} \right) = Z^l \left(\text{diagram 2} \right) = \exp(-a_1 - a_2) Z^l \left(\text{diagram 3} \right).$$

Now, let X be the element of $\mathcal{A}(M_{1,2})$ such that

$$Z^l \left(\text{diagram 4} \right) = 1_{\text{diagram 5}} \times X.$$

Here X is simply included in $\mathcal{A}(M_{1,2,3,4})$, with the univalent vertices staying only on the first and second component, on the left.

According to Proposition 2.11, we have

$$Z^l \left(\text{diagram 6} \right) = \delta(R_{\{1,2\}\{3,4\}}^{-2}).$$

We know that the elements of $\delta(\mathcal{A}(M_{\{\{1,2\}\{3,4\}\}}))$ commute with the elements of $\mathcal{A}(M_{\{1,2\}})$, so we have

$$Z^l \left(\text{diagram 7} \right) = 1_{\text{diagram 5}} \times \delta(R_{\{1,2\}\{3,4\}}^{-2}) \times X.$$

But

$$1_{\text{diagram 5}} \times \delta(R_{\{1,2\}\{3,4\}}^{-2}) = \delta(1_{\cap} \times R^{-2}) = \delta(\exp(a_{\{1,2\}})) = \exp(\delta(a_{\{1,2\}}))$$

since we can easily check that the product by 1_{\cap} is a morphism of algebras from $\mathcal{A}(M_{\{1,2\}})$ to $\mathcal{A}(M_{\{1\}})$, and the map δ from $\mathcal{A}(M_{\{1\}})$ to $\mathcal{A}(M_{\{1,2\}})$ is also a morphism of algebras.

Now we can conclude: we have

$$Z^l \left(\text{diagram 1} \right) = 1_{\text{diagram 5}} \times R_{3,4}^4 \times X.$$

Then, we just simplify by X and find

$$R^4 = \exp(-a_1 - a_2) \exp(\delta(a_{\{1,2\}})).$$

The degree zero part of R is 1 since R is grouplike, so we can apply the development of the fourth root to obtain the result:

$$R = \exp\left(\frac{\delta(a_{\{1,2\}}) - a_1 - a_2}{4}\right).$$

□

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